Anomalous Zero Sound

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Motivation

- A wide range of anomaly-induced effects are being intensively discussed:
  - Chiral magnetic effect
  - Chiral charge separation
  - Chiral spiral effect
  - Chiral magnetic wave effect
- Effective description of these phenomena is available both on gauge and gravity sides of the AdS/CFT correspondence.
- Collective fluctuation modes
  - Zero sound
  - First sound

  have also been widely studied holographically and field-theoretically.

Here we explore the consequences of bringing the idea of anomaly together with the idea of collective fluctuation modes

Landau Theory - a Reminder

The Landau theory of a Fermi liquid allows us to predict a lot of interesting collective phenomena in quantum liquids.

Its main tool is the **kinetic equation:**

$$\frac{\partial n(\mathbf{p},\mathbf{x})}{\partial t}+\frac{\partial \epsilon(\mathbf{p},\mathbf{x})}{\partial \mathbf{p}}\frac{\partial n(\mathbf{p},\mathbf{x})}{\partial \mathbf{x}}-\frac{\partial \epsilon(\mathbf{p},\mathbf{x})}{\partial \mathbf{x}}\frac{\partial n(\mathbf{p},\mathbf{x})}{\partial \mathbf{p}}=0$$

$n(\mathbf{p},\mathbf{x})$ is the particle density, $\epsilon(\mathbf{p},\mathbf{x})$ is the energy density; collisionless limit taken.
Its features are:

- The whole system “contributes energy” into the given mode
- The natural basis for the Hilbert space of excited states is the quasiparticle basis

The kinetic equation is the semiclassical limit of the evolution equation for the density operator n

$$\frac{\partial n}{\partial t}=\frac{i}{\hbar}[n,H],$$

classical Poisson bracket of $\{\epsilon, n\}$ emerging from the commutator $[n, H]$. Alternatively it is understood as the classical continuity equation

$$\frac{\partial n(\mathbf{x})}{\partial t}+\frac{\partial \mathbf{j}(\mathbf{x})}{\partial \mathbf{x}}=0$$

Fermi Liquids

To describe a quantum liquid we assume that

- The energy in the **p** mode "feels" the fluctuations of the rest of the liquid.
- There exist a spatially homogeneous equilibrium energy and density distributions $\epsilon_0(\mathbf{p}), n_0(\mathbf{p})$ of interacting quasiparticles
- The interaction strength is not necessarily small.

The energy fluctuation $\delta\epsilon(\mathbf{p}, \mathbf{x})$ is a function of both momenta and coordinates

$$\epsilon(\mathbf{p}, \mathbf{x}) = \epsilon_0(\mathbf{p}) + \delta\epsilon(\mathbf{p}, \mathbf{x}),$$

The density fluctuations δn $n(\mathbf{p}, \mathbf{x}) = n_0(\mathbf{p}) + \delta n(\mathbf{p}, \mathbf{x})$, are related to energy fluctuations via the "interaction kernel" $f(\mathbf{p}, \mathbf{p}')$

$$\delta\epsilon(\mathbf{p}, \mathbf{x}) = \int \frac{d^3p'}{(2\pi\hbar)^3} f(\mathbf{p}, \mathbf{p}')\delta n(\mathbf{p}, \mathbf{x}),$$

which becomes

$$\delta\epsilon(\mathbf{p}, \mathbf{x}) = \int d^2\Omega\, F(\theta, \theta')\delta n(\theta', \mathbf{x}),$$

where f is a matrix that includes spin-spin interaction formfactor $\sim \sigma\sigma'$; here we limit ourselves to the class of interactions with $f \sim \hat{1}$.

Density Fluctuations

The linearized kinetic equation for fluctuations becomes

$$\frac{\partial \delta n}{\partial t} + \frac{\partial \epsilon_0}{\partial \mathbf{p}} \frac{\partial \delta n}{\partial \mathbf{x}} - \frac{\partial \delta \epsilon}{\partial \mathbf{x}} \frac{\partial n_0}{\partial \mathbf{p}} = 0.$$

Now, consider a flat-wave fluctuation of the particle density

$$\delta n = \delta(\epsilon - \epsilon_F)\nu(\theta, \phi)e^{i(\omega t - \mathbf{kr})}.$$

Taking into account that $\frac{\partial n_0(\mathbf{p},\mathbf{x})}{\partial \mathbf{p}} = -\mathbf{v}_F \delta(\epsilon - \epsilon_F)$, we get the following integral equation

$$(\omega - \mathbf{kv}) = \mathbf{kv} \int \frac{d^2\Omega'}{4\pi} F(\theta, \theta', \phi, \phi')\nu(\theta'),$$

where $d^2\Omega = \sin\theta d\theta d\phi$. Since $\mathbf{kv} = kv_F \cos\theta$, introducing $s \equiv \frac{\omega}{kv_F}$, we get

$$(s - \cos\theta) = \cos\theta \int \frac{\sin\theta' d\theta'}{2} \nu(\theta')F(\theta, \theta').$$

Zero Sound

Let us limit ourselves with the the zeroth and the first harmonics of F

$$F = F_0 + F_1 \left(\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi') \right).$$

This will mean that a solution to it must be sought in the form

$$\nu = (C_0 + C_1 \cos\theta) \frac{\cos\theta}{s - \cos\theta},$$

where C_0, C_1 are some constants which are subject to conditions

$$C_0 = F_0 \int \frac{\sin\theta d\theta}{2} (C_0 + C_1 \cos\theta) \frac{\cos\theta}{s - \cos\theta}, \quad C_1 = F_1 \int \frac{\sin\theta d\theta}{2} \cos\theta (C_0 + C_1 \cos\theta) \frac{\cos\theta}{s - \cos\theta}.$$

This system of linear equations upon C_0, C_1 must have a zero determinant to be solvable

$$\left(\frac{1}{2} s \log\left(\frac{s+1}{s-1} \right) - 1 \right) \left(F_0(F_1 + 3) + 3F_1 s^2 \right) = F_1 + 3.$$

For small F_0, F_1 we expand the equation around $s = 1$ and get the **dispersion law**

$$s = 1 + 2e^{-\frac{2(F_0(F_1+3)+4F_1+3)}{F_0(F_1+3)+3F_1}},$$

whereas for the strongly-interacting case we get

$$s = \frac{1}{\sqrt{3}} \sqrt{F_0 + \frac{3F_1}{5} + \frac{F_0 F_1}{3}}.$$

Idea of the Anomalous Zero Sound

Zero sound – not a **single quasiparticle** excitation but a **collective excitation** of the Fermi surface.

- In presence of both left and right fermions zero sound is twice degenerate
- Consider current modification due to anomaly
- This will mix left and right, altering the dispersion law and lifting the degeneracy
- This can be done either via Fermi liquid theory or holography.

Now we proceed to the modification of zero sound due to an anomaly in the magnetic field.

Anomaly and Chirality

Son and Yamamoto *[Son, Yamamoto 2012]* suggested that in theories with anomalies the current be supplied with two extra terms

$$\mathbf{j}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\left[-\epsilon(\mathbf{p},\mathbf{x})\frac{\partial n(\mathbf{p},\mathbf{x})}{\partial p} - \left(\vec{\Omega}\frac{\partial n(\mathbf{p},\mathbf{x})}{\partial \mathbf{p}}\right)\epsilon(\mathbf{p},\mathbf{x})\mathbf{B} - \epsilon(\mathbf{p},\mathbf{x})\left(\vec{\Omega}\times\frac{\partial n(\mathbf{p},\mathbf{x})}{\partial \mathbf{x}}\right)\right].$$

here $\vec{\Omega}$ is the "dual" magnetic field strength. The extra current leads to several effects:

- Chiral magnetic effect – current generation along the **B**-field

$$\mathbf{j} = \frac{\mu_A \mathbf{B}}{2\pi^2}.$$

  This phenomenon was realized in the holographic approach to QCD in several different ways *Yee 2009, Rebhan et al 2009, Gorsky et al 2010*.

- Another effect this current leads to is the chiral magnetic wave. The essence of this effect is propagation of a chirality wave through a medium in a magnetic field $\mathbf{B} = (0, 0, B)$

$$n_A \sim e^{i(\omega t - kz)}.$$

Anomalous Current

The modification of the current allows us to include the anomaly-driven dynamics into the kinetic equation. Consider a Fermi liquid with right and left quasiparticles, with densities n_R and n_L. Then

$$\mathbf{j}_{R,L}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\left[-\epsilon_{R,L}(\mathbf{p},\mathbf{x})\frac{\partial n_{R,L}(\mathbf{p},\mathbf{x})}{\partial p} - \left(\vec{\Omega}_{R,L}\frac{\partial n_{R,L}(\mathbf{p},\mathbf{x})}{\partial \mathbf{p}}\right)\epsilon_{R,L}(\mathbf{p},\mathbf{x})\mathbf{B} - \right.$$
$$\left. -\,\epsilon_{R,L}(\mathbf{p},\mathbf{x})\left(\vec{\Omega}_{R,L}\times\frac{\partial n_{R,L}(\mathbf{p},\mathbf{x})}{\partial \mathbf{x}}\right)\right],$$

This will lead to the following effects:

- Left and right oscillations become now coupled
- Degeneracy in the dispersion law lifted
- This phenomenon will be called by us **"anomalous zero sound"**.

Relativistic vs. Nonrelativistic Fermions

How can one combine

- non-relativistic physics of quasiparticles living on a Fermi surface
- an explicitly relativistic extra current by Son-Yamamoto.

Extra current terms are derived from the Berry phase – monopole at the origin (momentum space!) is required.
This makes sense for massless particles only.
It would be better to derive an analog of the anomalous extra current for a massive theory.
The anomalous contribution by Son and Yamamoto exists both for $m = 0$ and $m \neq 0$ – therefore we keep the monopole at the origin of the momentum space for the massive particles as well.

The "dual" magnetic field $\vec{\Omega}$ is

$$\vec{\Omega}_{R,L} = \pm \frac{\mathbf{n}}{2p_F^2},$$

where p_F is Fermi momentum, $\mathbf{n}$ is the unit normal vector , $d\mathbf{S}$ is the integration measure on the Fermi surface.

The extra contributions in the kinetic equation $\frac{\partial n(\mathbf{x})}{\partial t} + \frac{\partial \mathbf{j}(\mathbf{x})}{\partial \mathbf{x}}$ that come from $\left(\vec{\Omega}\frac{\partial n(\mathbf{p},\mathbf{x})}{\partial \mathbf{p}}\right)\epsilon(\mathbf{p},\mathbf{x})\mathbf{B}$ are the following terms

$$(\nabla \mathbf{j})_1 = \nabla_i \left(\Omega^j \frac{\partial \overset{\downarrow}{n}(\mathbf{p},\mathbf{x})}{\partial p^j}\right)\epsilon(\mathbf{p},\mathbf{x})B^i,$$

$$(\nabla \mathbf{j})_2 = \nabla_i \left(\Omega^j \frac{\partial n(\mathbf{p},\mathbf{x})}{\partial p^j}\right)\overset{\downarrow}{\epsilon}(\mathbf{p},\mathbf{x})B^i.$$

The contribution of $\epsilon(\mathbf{p},\mathbf{x})\left(\vec{\Omega}\times\frac{\partial n(\mathbf{p},\mathbf{x})}{\partial \mathbf{x}}\right)$ is zero.

Anomaly and Density Waves

The density waves will be sought in the same form as prescribed by Landau theory for zero sound.

$$\delta n_{R,L} = \delta(\epsilon - \epsilon_F)\nu_{R,L}(\theta, \phi)e^{i(\omega t - \mathbf{kr})}.$$

we model the energy fluctuations as

$$\delta\epsilon_R = \int (F_S\nu_R + F_A\nu_L)\frac{\sin\theta' d\theta'}{2}, \quad \delta\epsilon_L = \int (F_A\nu_R + F_S\nu_L)\frac{\sin\theta' d\theta'}{2}$$

The two contributions in questions then become

$$(\nabla \mathbf{j}_R)_1 = \frac{ik\nu_R(\theta)\delta(\epsilon - \epsilon_F)Bv_F}{4p_F^2},$$

$$(\nabla \mathbf{j}_R)_2 = \frac{ik\nu(\theta)\delta(\epsilon - \epsilon_F)Bv_F}{2p_F^2}\int \left(F_S\nu_R(\theta') + F_A\nu_L(\theta')\right)\frac{\sin\theta' d\theta'}{2}.$$

the equation is transformed into

$$(s - \cos\theta - b)\nu_R = (b + \cos\theta)(F_S C_R + F_A C_L),$$
$$(s - \cos\theta - b)\nu_L = (b + \cos\theta)(F_A C_R + F_S C_L),$$

where a convenient dimensionless parameter

$$b \equiv \frac{Bv_F}{2p_F^2}$$

has been introduced. Solving this system with regard to $\nu_{R,L}$ and imposing the conditions $\int \frac{\sin\theta' d\theta'}{2}\nu_R(C_R, C_L) = C_R$, $\int \frac{\sin\theta' d\theta'}{2}\nu_L(C_R, C_L) = C_L$ we get thus a homogeneous system upon the coefficients $C_{R,L}$. Requiring its determinant to be zero we obtain the dispersion laws.

Dispersion

For small coupling we see that the degeneracy has indeed been lifted

$$\omega(k) = 1 \pm b + e^{-\frac{2}{F_S(1\mp b)}}.$$

At large coupling we get

$$\omega(k) = \sqrt{\frac{F_S}{3}\left(1 - \frac{F_A^2 - F_S^2}{2F_S}\right)}$$

Notice that in both cases the field B has been held arbitrary; the only perturbative expansion used was the expansion in small or large $F_{S,A}$

These dispersion laws are the main result of the field-theoretical part of this work. We interpret this situation in terms of two types of zero sound, axial and vector one propagating through a medium of left and right fermions with zero net chirality, which interact due to the presence of the B field.

Zero Sound and Holography

The oscillation mode we have described so far is known as the zero sound mode. It is well known in the holographic setup:

- Holographic zero sound first realized in *[Karch, Son, Starinets 2008]* at zero temperature.
- At finite temperature holographic zero sound constructed in *[Davison, Starinets 2011]* (previous talk)
- Holography reproduces the dispersion law for the large F_0, F_1, rather than the weakly-coupled regime.
- In particular, even the quantum attenuation contribution (the absorptive part in the dispersion law not shown here) is reproduced form holography as well *[Karch, Son, Starinets 2008]* .

Collective modes

There exist different collective modes relevant to our subject which can be calculated from holography

mode	oscillating quantity	holographic recipe	reference
Sound	density	$\langle TT \rangle$	[Policastro, Son, Starinets hep-th/0210220]
Zero sound	Fermi sphere	$\langle JJ \rangle$	[Karch, Son, Starinets 0806.3796]
Chiral spiral wave	axial condensate	$\langle XX \rangle$	[Kharzeev, Yee 1109.0533]
Chiral magnetic wave	axial density	$\langle JJ \rangle\|_{\mu=0}$	[Kharzeev, Yee 1012.6026]

Collective modes manifest themselves as poles in the correlators.

NB Zero sound is to be seen at small temperatures and large chemicall potentials; first sound - vice versa.

Holography Setup

The normal zero sound (i.e. without the anomaly) in holography has been discovered by Karch, Son and Starinets. It can be easily modified to account for the anomalous effects, resident in the Chern-Simons term on the $D7$-brane. Consider the standard case of a $D7$-brane embedding action into the geometry of N_c $D3$-branes:

	0	1	2	3	4	5	6	7	8	9
$D3$	+	+	+	+						
$D7$	+	+	+	+	+	+	+	+		

Now take there are two of them, one corresponding to the left modes and the other to the left ones. The full action is then

$$S = S^L_{DBI} + S^R_{DBI} + S^L_{CS} - S^R_{CS}$$

where the Dirac–Born–Infeld action for either left or right fermions is

$$S^{L,R}_{DBI} = -T_{D7} \int d^8\xi \sqrt{-\det\left(g_{ij} + 2\pi\alpha' F^{L,R}_{ij}\right)}$$

which in the metric

$$ds^2 = \frac{r^2}{R^2} dx^2_{1,3} + \frac{R^2}{r^2}\left(dr^2 + r^2 d\Omega_5^2\right)$$

in presence of the vector-potentials $A^{L,R}_0(r)$, and a constant field $F^R_{12} = F^L_{12} = B$ becomes

$$S = -\mathcal{N} V_0 \int r^3 \sqrt{1 - \left(\frac{\partial A^{L,R}_0}{\partial r}\right)^2 + B^2}$$

(here we have included the $2\pi\alpha'$ into the definitions of the fields). The Chern–Simons part is

$$S_{CS} = -\frac{N_c}{24\pi^2} \int dr d^4x \epsilon^{\mu\nu\lambda\rho\sigma} A_\mu F_{\nu\lambda} F_{\rho\sigma}.$$

Boundary Conditions

We can solve the classical equations of motion for the L and R modes independently for $\mu_L = \mu_R \equiv \mu$ by the Ansatz

$$A_{0,cl}^{L,R\prime}(r) = \frac{d\sqrt{1+B^2}}{\sqrt{r^6+d^2}}$$

where the integration constant $d \sim \mu^{\frac{1}{3}}$. Considering fluctuations

$$A_0^{L,R}(r) = A_{0,cl}^{L,R}(r) + a_0^{L,R}(r,x)$$

we get

$$\begin{aligned}
\partial_z\left(\frac{f^3}{z}a_0^{V\prime}\right) - \frac{qf}{z}(\omega a_3^V + qa_0^V) + iba_3^{A\prime} &= 0,\\
\partial_z\left((\frac{f}{z}a_3^{V\prime}\right) + \frac{\omega f}{z}(\omega a_3^V + qa_0^V) - iba_0^{A\prime} &= 0,\\
\partial_z\left((\frac{f^3}{z}a_0^{L\prime}\right) - \frac{qf}{z}(\omega a_3^A + qa_0^A) - iba_3^{V\prime} &= 0,\\
\partial_z\left((\frac{f}{z}a_3^{A\prime}\right) + \frac{\omega f}{z}(\omega a_3^A + qa_0^A) + iba_0^{V\prime} &= 0,
\end{aligned}$$

the remnant gauge fixing condition being $f^2\omega a_0^{V,A\prime} + qa_3^{V,A\prime} = 0$, where we have switched to the variable $z \equiv \frac{1}{r}$ and introduced the function $f = \frac{\sqrt{1+d^2z^6}}{\sqrt{1+B^2}}$, and absorbed the normalization factor into $b \sim B$.

We introduce two gauge-invariant field strengths

$$E^V = \omega a_3^V + q a_0^V$$
$$E^A = \omega a_3^A + q a_0^A$$

and construct a complex variable out of them

$$E = E^V + iE^A,$$

now representing the dynamics in the chiral plane, thus we can finally write down the equation upon the complex variable E

$$E''(z) + E'(z)\left(\frac{f'(z)\left(3q^2 - \omega^2 f(z)^2\right)}{f(z)\left(q^2 - \omega^2 f(z)^2\right)} - \frac{1}{z}\right) + \frac{E(z)\left(\omega^2 f(z)^2 - q^2\right)}{f(z)^2} + \frac{ibq\omega z E'(z)}{f(z)\left(q^2 - \omega^2 f(z)^2\right)} = 0$$

This equation is then studied numerically. It is subject to the boundary equations

$$\begin{cases} E(z)|_{z\to 0} = 0 \\ \partial_z(zE(z))|_{z\to\infty} = i\omega z E(z)|_{z\to\infty} \end{cases}$$

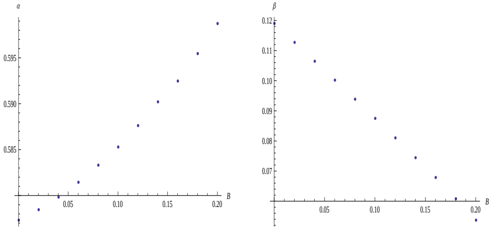
The first of these conditions is the normal Dirichlet boundary condition on the boundary; the second one is the infalling wave condition on the horizon.

Dispersion Law

These two conditions fix eigenvalues of this equation

$$\omega = \omega(k) = \alpha(B)k - i\beta(B)k^2.$$

The numerical dependencies of the phase velocity of the wave α and the diffusion coefficient β on the magnetic field are given in the figures here:



One can readily see that the α dependence starts with a correct $1/\sqrt{3}$ value, and then changes linearly. This is surely the Chern-Simons contributions, since the DBI could have contributed only in the order of $\sim B^2$.

Thus we have produced in this section the dispersion relation for the chiral anomalous zero sound in holography.

Mind the gap

We dealt with a gapless longitudinal mode in external field. As shown by *[Goykhman, Parnachev and Zaanen 2012]* it will actually acquire a gap. In their work:

- Zero sound from holography for systems in a magnetic field at a zero temperature and finite chemical potential was considered.
- External magnetic field-driven mixing between the bulk vector longitudinal mode oscillations E discussed above and their transverse counterpart (not discussed here) was found
- Mass generation for the originally massless longitudinal E mode due to mixing arises. The mixing-induced mass squared term is proportional to B^2.
- No pole for the transverse mode; thus no (quasi)particle corresponds to it on the boundary.

Mixing and the Mass

How can be the results by GPZ combined with ours?

- Transverse-longitudinal mixing present in our case as well
- To isolate effects of the anomaly we switch it off
- In the leading order our effect modifying the wave velocity, and Goykhman-Parnachev-Zaanen effect generating the gap – will be simply superposed as independent effects
- Anyway the leading-order effect at small B is our anomalous mixing

At large B some new effects might arise, involving both the physics due to gap and due to mixing. This must be left for future investigations.

Results Summary

- Anomaly modification of zero sound obtained
- Dispersion laws calculated for weakly and strongly coupled anomalous zero sound from the kinetic equation
- Holographic setting for the anomalous zero sound provided; dispersion law calculated from holography

At the holographic side we have generalized the consideration of *[Goykhman et al. 2012]* taking into account the Chern-Simons term and have found the clear-cut manifestation of the anomalous contributions at the strong coupling.

Discussion

Further questions:

- Transverse modes
- Spin wave effects
- Spin wave – sound wave interaction

Our consideration has a lot in common with the **chiral magnetic wave**, however in that case the the modes in plasma imply high temperature and small μ while in our case we discuss the dense matter at very small or vanishing temperature.