On UV and IR geometry of supersymmetric gauge theories

Andrei Marshakov
Lebedev Institute, ITEP & NRU-HSE, Moscow

HATCH, Moscow, June 2012

- Applications to technicolor, condensed matter and hadrons - neither of them directly;

- Instead - an application of holographic principle, when something can be verified: gravity duals for the SUSY gauge theories (large N, but qualitatively correct even for $N = 2, 3, \ldots$);

- AdS/CFT correspondence of $\mathcal{N} = 4$ SYM with string theory on $AdS_5 \times S^5$ $\rightarrow$ M-theory duals for SYM;

- Integrable system as a tool for the tests/checks. Exact results.

AdS/CFT conjecture:

- $\mathcal{N} = 4$ SYM with the $SU(N)$ gauge group in $\frac{1}{N}$ -expansion at fixed 't Hooft coupling $\lambda = g^2_{YM} N = g_s N$;

- At $\lambda \gg 1$ is dual to the IIB string theory in $AdS_5 \times S^5$ with radius $R = \lambda^{1/4}\sqrt{\alpha'}$. Two different ($\lambda$- versus $\frac{1}{\sqrt{\lambda}}$- expansions compared by considering *classical* string solutions and "large" operators in SYM;

- M-theory dual: stack of M5-branes on a torus (with single puncture) - solution to 11-dimensional supergravity.

Holography: boundary of 10-dim manifold with the metric

$$ds^2\Big|_{10} = ds^2\Big|_{AdS_5} + ds^2\Big|_{S^5} \tag{1}$$

where the Lobachevsky (or hyperbolic) part is

$$ds^2\Big|_{AdS_5} = \frac{dx_\mu dx^\mu + dr^2}{r^2} \tag{2}$$

Scale invariance (conformal group in 4d) - no nontrivial IR behaivior (far from horizon at $r \to 0$).

Nontrivial IR theory: breaking (under control) $\mathcal{N} = 4$ (e.g. in UV) to $\mathcal{N} = 2$: what happens in terms of gravity duals?

Plenty of possible scenario: just about one of the ways ...

M-theory compactifications $ds^2\Big|_{11} = ds^2\Big|_7 + ds^2\Big|_4$:

$$ds^2\Big|_7 = ds^2\Big|_5 + ds^2\Big|_{\Sigma_0} \underset{r\to 0}{=} \; ds^2\Big|_{AdS_7} \tag{3}$$

where (at $r \to 0$ impose $f(r), \phi(r) \sim -\log r$)

$$\begin{aligned} ds^2\Big|_5 &= e^{2f(r)}\left(dx_\mu dx^\mu + dr^2\right) \underset{r\to 0}{=} \; ds^2\Big|_{AdS_5} \\ ds^2\Big|_{\Sigma_0} &= e^{2\phi(r)}\frac{dz d\bar{z}}{(\mathrm{Im} z)^2} \end{aligned} \tag{4}$$

corresponds to wrapping of stack of M5-branes onto the Riemann surface Σ_0 of high genus $g_0 > 1$.

Riemann surface - factor of the Lobachevsky plane over discrete group.

- Stack of N M5 brane wrapped onto Σ_0 - UV (?) curve (6-dimenional UV completion), low genera with punctures - non-compact M5 branes, intersecting Σ_0 at the punctures;

- UV geometry: moduli spaces of flat connections on Σ_0 - turning on Wilson lines for the vector fields in supergravity solutions;

- IR geometry: infinitesimally - deformation of Σ_0, globally - N-sheet cover of Σ_0 for the stack of N M5-branes.

$SL(N,\mathbb{C})$-valued flat connections on $\Sigma_0 = \Sigma_{g_0,n}$:

$$\begin{aligned} \dim_{\mathbb{C}}(\mathcal{M}_{sl_N}(\Sigma_0)) = 2(g_0-1)(N^2-1) + n(N^2-N) = \\ \underset{g_0=0}{=} \; 2\left(n\frac{N(N-1)}{2} - N^2 + 1\right) \end{aligned} \tag{5}$$

For the $N=2$ case ($SL(2,\mathbb{C})$ connections) this is dimension of $T^*\mathcal{T}(\Sigma_0)$ - cotangent to the Teichmüller space of Σ_0.

- $\Sigma_0 = \Sigma_{g_0,n}$ itself defines UV theory: $\mathcal{T}(\Sigma_0)$ - the space of UV couplings;

- Deformations of $\Sigma_0 = \Sigma_{g_0,n}$ - IR theory: co-ordinates in T^* correspond to the vacuum condensates.

Flat connections $\to$ Teichmüller: Σ_0 with a two-differential $t = t(z)dz^2$, at each puncture

$$t(z) \underset{z \to z_j}{=} \frac{\Delta_j}{(z-z_j)^2} + \frac{u_j}{z-z_j} + \ldots, \qquad j = 1, \ldots, n \tag{6}$$

where "masses" Δ_j are fixed. Deformation

$$\begin{gathered} \delta t = \sum_{j=1}^{n} \delta u_j h_j + \sum_{k=1}^{3g_0-3} \delta y_k h_k \\ h_j = \frac{dz^2}{z-z_j} + \ldots, \qquad h_k = \text{holomorphic} \end{gathered} \tag{7}$$

with $u_j \sim \langle \mathrm{Tr}\Phi_j^2 \rangle$ being scalar condensates.

Effective IR theory - prepotential: SW curve Σ (covers Σ_0 twice for $N=2$)

$$x^2 = t \tag{8}$$

The genus g of Σ (from the Riemann-Hurwitz formula)

$$g = 4g_0 - 3 + n = \dim_{\mathbb{C}} \mathcal{T}(\Sigma_0) + g_0 \tag{9}$$

Define ($a_i = \oint_{A_i} dS$, $i\pi\tau_j^{(0)} = \log q_j = \int_{B_j^{(0)}} d\Omega^{(0)}$)

$$\begin{gathered} a_i^D = \frac{\partial \mathcal{F}}{\partial a_i} = \oint_{B_i} dS, \quad i = 1, \ldots, g \\ \frac{\partial \mathcal{F}}{\partial \tau_j^{(0)}} = \int_{A_j^{(0)}} \frac{dS}{d\Omega^{(0)}} dS \end{gathered} \tag{10}$$

where $dS = xdz$ is the SW differential (here $A, B \in H_1(\Sigma)$ while $A^{(0)}, B^{(0)} \in H_1(\Sigma_0)$.

The prepotential $\mathcal{F}(\mathbf{a}, \boldsymbol{\tau}^{(0)})$ defines effective action for the *quiver* gauge theory with the gauge group $G = \bigotimes_{r=1}^{L} G_r$ factors (e.g. $G_r = SU(2)$, $\forall r$)

$$L = 3g_0 - 3 + n = \dim_{\mathbb{C}} \mathcal{T}(\Sigma_0) \tag{11}$$

For consistency at each $r = 1, \ldots, L$

$$\beta_r = 2N_c^{(r)} - N_f^{(r)} - \sum_{r' : \langle r, r' \rangle \neq 0} N_c^{(r')} N_{bf}^{(r,r')} \geq 0 \tag{12}$$

restricted to fundamental (f) and bi-fundamental (bf) matter.

In practice $g_0 = 0, 1$ - very degenerate holographic picture. ($g_0 > 1$ theories cannot be formulated in weak-coupling regime - lack of Lagrangian description?).

In the simplest example $g_0 = 0$, $n = 4$, $L = 3$ and

$$x^2 = \sum_{j=1}^{4} \frac{u_j}{z - z_j} \qquad (13)$$

is SW curve for $SU(2)$ (superconformal) theory with $N_f = 4$ massless multiplets and the only UV coupling

$$i\pi\tau_0 = \int_{B_0} d\Omega^{(0)} \underset{\{z_j = 0, q, 1, \infty\}}{=} \int_1^q \frac{dz}{z} = \log q \qquad (14)$$

Then

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial a} &= \frac{1}{2\pi i} \oint_B x dz = a\tau \\ q\frac{\partial \mathcal{F}}{\partial q} &= \tfrac{1}{2} \int_{A_0} \frac{dS}{d \log z} dS = \tfrac{1}{2} \int_{A_0} x^2 z dz \end{aligned} \qquad (15)$$

It means, that for prepotential $\mathcal{F} = \frac{1}{2}\tau a^2$ ($\tau \equiv \tau_{IR} \neq \tau_0$)

$$\frac{d\tau}{d\tau_0} = \frac{1}{\alpha^2(\tau)(q-1)}$$
$$\alpha(\tau) = \frac{1}{2\pi}\oint_A \frac{dz}{\sqrt{z(z-1)(z-q)}} = \theta_3^2(0|\tau) \quad (16)$$

or

$$\frac{d\tau_0}{d\tau} = \alpha^2(\tau)(q-1) = -\frac{1}{\pi^2}\theta_4^4(0|\tau) \quad (17)$$

which integrates to Zamolodchikov's renormalization formula $\exp(i\pi\tau_0) = q = \theta_2(0|\tau)^4/\theta_3(0|\tau)^4$, after using an identity

$$\theta_2''(0|\tau)\theta_3(0|\tau) - \theta_3''(0|\tau)\theta_2(0|\tau) = -\theta_4(0|\tau)^4\theta_2(0|\tau)\theta_3(0|\tau)$$

- In the same spirit works for the “weakly-coupled” quivers;

- For $g_0 > 1$ problems with the geometric formulation in UV;

- Exact formulation for the holographically predicted strongly coupled superconformal theories - to be yet understood.

List: $E_{6,7,8}$ superconformal points, Argyres-Seiberg duality, generic $\bigotimes_{r=1}^{L} SU(N_r)$ Gaiotto quivers, . . .

Conclusions

- Nontrivial test of the holographic correspondence of $\mathcal{N}=4$ SYM with $AdS_5 \times S_5$ gravity;

- Gravity duals for $\mathcal{N}=2$ theories predict geometry in UV;

- Link to IR by study geometry of complex curves and integrable systems;

- Tau-functions for the quiver theories at weak coupling. Strong coupling regime??