Emergent Lorentz Invariance: Holographic Description.

Grigory Bednik
in collaboration with Sergey Sibiryakov

Institute for Nuclear Research of the Russian Academy of Sciences

June 2012

Publication is in preparation

Introduction

- We are interested in systems where Lorentz invariance emerges at low energies. Our goal is to provide the holographic description of this phenomenon.
- We consider $(d+1)$ - dimensional space interpolating between AdS in the bulk and Lifshitz near the boundary (proposed by Kachru, Liu, Mulligan, 2008)
- We calculate correlators of the scalar field in this space
- and dispersion relations of discrete excitations of the field (if the space is surrounded by branes)

Historical background. A model of Chadha and Nielsen.

- A toy model with Lorentz-invariance violation (Chadha, Nielsen 1982):

$$L = \frac{F_{0i}^2}{2} - \frac{c_\gamma^2 F_{ij}^2}{4} + \bar{\psi}\left(i\gamma^0 D_0 + i c_e \gamma^i D_i - m\right)\psi$$

- RG evolution. $c_\gamma - c_e$ vanishes in the IR.
- This vanishing, i.e. Lorentz-invariance emergence is more efficient at strong coupling
- This property holds in morte general theories.
  - Yengo, Russo, Serone (2009)
  - Giudice, Raidal, Strumia (2010)
- At strong coupling it is natural to use holography.

The space

- This space is formed by massive vector A_M interacting with gravity.
- The form of the metric

$$ds^2 = L^2(-\frac{f^2 dt^2}{u^2} + \frac{g^2 du^2}{u^2} + \frac{1}{u^2}\sum_{i=1}^{d-1} dx_i^2),$$

- This space is formed by massive vector A_M interacting with gravity.

$$S = \int d^{d+1}x\sqrt{-g}(R-2\Lambda) - \frac{1}{4}\int d^{d+1}x\sqrt{g}(F_{MN}F^{MN} + 2m^2 A_M A^M).$$

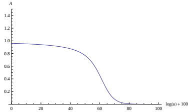
- Fixed points:

$$f = u^{1-z}, \qquad g = \mathrm{const}$$

Fixed points and our solution

- The fixed points:
  - $z = 1$ - anti-de Sitter space
  - $z \neq 1$ and $\tilde{z} = \frac{(d-1)^2}{z}$ - Lifshitz space.
- The Lifshitz fixed point is stable when $1 < z < d - 1$.
- Our space approaches to:
  - *AdS* far from the boundary
  - Lifshitz near the boundary

$A_M^2(u)$



Probe scalar field

- We consider probe scalar field in the space interpolating between *AdS* and Lifshitz

$$S = \frac{1}{2} \int d^{d+1}x \sqrt{g} \left(g^{MN} \partial_M \Phi \partial_N \Phi + M^2 \Phi^2 \right)$$

- We calculate:
  - the correlator of the dual theory,
  - dispersion relations of discrete excitations

  in the limits of small and large frequencies

Correlator at small w, k

- We need solution in AdS decreasing far from the boundary.
- In AdS area $\Phi = u^{\frac{d}{2}} K_{\frac{d}{2}}(\sqrt{w^2+k^2}u)$
- Near the boundary $\Phi = \phi_0 + \phi_1 + \dots$

$$\begin{aligned}
\phi_0 &= \mathrm{const} \\
\phi_1 &= \int_0^u \frac{du u^{d-1} g}{f} \left(\phi_0 \int_{u_1}^u \frac{fg}{u^{d-1}} (\frac{w^2}{f^2} + k^2) du + C \right)
\end{aligned}$$

- We expand Φ in AdS, match them, find C.
- $\langle O_\Phi(k,w), O_\Phi(-k,-w)\rangle =$ $\Phi(u,-k,-w) g^{uu} \sqrt{g} \partial_u \Phi(u,k,w) \mid_{u_1 \to 0}^{\infty} =$

$$\begin{cases} L^2 g_0^2 (w^2+k^2)^{3/2}, & d=3; \\ -\frac{L^3 g_0^3 (w^2+k^2)^2}{4} \ln(g_0 u_2 \sqrt{w^2+k^2}), & d=4. \end{cases}$$

Brane problem at small w, k with Dirichlet boundary conditions at the boundary

- Let us border the space by the brane at large $u = u_0$
- We consider discrete excitations of the scalar field satisfying to Neumann boundary conditions at u_0 and Dirichlet boundary conditions at $u = 0$.
- At $d = 3$
  - At the "domain wall"side $\Phi = \phi_0 + \phi_1 + \ldots$ at $w, k << u_{DW}^{-1}$

$$\phi_1 = \int_{u_1}^{u} \frac{duu^2 g}{f} C,$$
$$\phi_2 = \int_{u_1}^{u} du \frac{u^2 g}{f} \int_{u_1}^{u} du \frac{fg}{u^2} (-\frac{w^2}{f^2} + k^2)\phi_1,$$

  - at the AdS side $\Phi_i = x\cos(x - x_0) - \sin(x - x_0)$
- We match them and get the equation for the masses:

$$(\int_{u_1}^{u_2} du \frac{u^2 g}{f} - \frac{u_2^3}{3}) = -\frac{\tan(u_0\sqrt{w^2 - k^2})}{(w^2 - k^2)^{3/2}}.$$

- Lorentz-invariance emerges!

Brane problem at small w, k with Neumann boundary conditions at the boundary

- The brane at large $u = u_0$
- Discrete excitations of the scalar field satisfying to Neumann boundary conditions at u_0 and Neumann boundary conditions at $u = 0$.
- At the "domain wall"side $\Phi = \phi_0 + \phi_1 + \ldots$,

$$\begin{aligned} \phi_0 &= \text{const} \\ \partial_u \phi_1 &= \frac{g u^{d-1}}{f} \int_{u_1}^{u} du \frac{fg}{u^{d-1}} \left(\frac{w^2}{f^2} + \frac{4\xi j^2 w^2}{m^2 L^2 f^2} - k^2 \right) \phi_0 \end{aligned}$$

- We can assume $w, k << u_0^{-1}$ and therefore we do not need to do the matching.
- The dispersion relation is: $w^2 = k^2 \frac{\int_0^{u_0} du \frac{fg}{u^{d-1}}}{\int_0^{u_0} du \frac{g}{u^{d-1} f}}$
- Lorentz-invariance emerges only at $d = 2$.

Correlator of the massive scalar

- We need solution in *AdS* decreasing far from the boundary.
- In *AdS* area $\Phi = u^{\frac{d}{2}} K_\nu(\sqrt{w^2+k^2}u), \quad \nu = \sqrt{\frac{d^2}{4} + M^2L^2}$
- Near the boundary $\Phi = \phi_0 + \phi_1 + \dots$
  - We cannot find ϕ_0, but we can express ϕ_1 through ϕ_0, φ_0:

$$\phi_1 = A(u)\phi_0 + a(u)\varphi_0$$

$$a = a_0 + \int_0^u du \frac{\phi_0^2\left(\frac{g^2w^2}{f^2}+g^2k^2\right)}{\phi_0\partial_u\varphi_0 - \varphi_0\partial_u\phi_0}$$

- The matching gives us $a_0 = \frac{\Gamma(-\nu+1)}{4^\nu\Gamma(\nu+1)}(\sqrt{w^2+k^2})^{2\nu}$
- The correlator is: $\langle O(k,w,u)O(-k,-w,u)\rangle =$ $\frac{L^{d-1}}{g_2} u_1^{2-d-z}\left(\frac{\partial_u\Phi_0}{\Phi_0} + \frac{(\Phi_0\partial_u\varphi_0 - \varphi_0\partial_u\Phi_0)}{\Phi_0^2}a\right) \propto (\sqrt{w^2+k^2})^{2\nu}$
- Lorentz-invariance emerges!

Corrections to the relations at small w, k.

- To find the relations with the corrections we need to match the solution in the "domain wall"area with the corrected solution in the AdS area.
- If Φ_i is the unperturbed solution, then the perturbed solution is $\Phi_i + a_0\varphi_i +$ *terms proportional to* $\delta f, \delta g$, *i.e.* $u^{-2\lambda}$, where $\lambda = -\frac{d}{2} + \frac{1}{2}\sqrt{(d-2)^2 + 4(mL)^2}$
- From the boundary conditions at large u_0 we find a_0.
- After the matching we find the correlator and the dispersion relation.
  - Correlator
    $$\langle O(k,w,u)O(-k,-w,u)\rangle \propto$$
    $$(\sqrt{w^2+k^2})^{2\nu}(1 + \mathrm{const}_1(\mathrm{w}^2 + \mathrm{k}^2) + \mathrm{const}_2\mathrm{w}^2)$$
  - Correction to the dispersion relation
    $$w^2 - k^2 = m^2 + \delta m^2$$
    $$\delta m^2 = -\frac{m_0(\mathrm{const}_1 + \mathrm{const}_2(\mathrm{w}^2 + \mathrm{k}^2) + \mathrm{const}_3\mathrm{w}^2)}{u_0\sqrt{w^2 - k^2}}$$

Semiclcassical treatment of the field equation

- By the change $\Phi = \left(\frac{gu^d}{f}\right)^{1/2}\psi$ the field equation transforms into Schrodinger-like equation:

$$-\partial_u^2\psi + V\psi = 0$$

- This equation can be solved via semiclassical approximation.

$$\Phi = \left(\frac{gu^d}{f}\right)^{1/2}\frac{1}{|V|^{1/4}}e^{\pm i\int_{u_t}^{u}\sqrt{|V|}du}$$

- The potential is divergent near the boundary. In case of moving away from the boundary V is decreasing and approaches to a negative constant $-w^2 + k^2$.
- Semiclassical approximation is not applicable near the boundary but it work in the area of negative V.
- At large w the semiclassical solution is applicable in the "domain wall"and AdS area.

Correlator at large w, k

- In th pure Lifshitz we know the solution at $z = 2$:

$$\Phi = u^{\frac{d+1}{2}-\sqrt{(\frac{d+1}{2})^2+M^2L^2}} e^{-\frac{wu^2}{2}}$$
$$U\left(\frac{k^2}{4w}+\frac{1}{2}-\frac{1}{2}\sqrt{(\frac{d+1}{2})^2+M^2L^2}, 1-\sqrt{(\frac{d+1}{2})^2+M^2L^2}, wu^2\right)$$

- It decreases rapidly, so we can assume that the correlator is the same as in pure Lifshitz.
- One can check it explicitly by matching this solution with the solution in the rest area which is found via semiclassical approximation.
- The correlator is:

$$\langle O_\Phi(k,w), O_\Phi(-k,-w)\rangle = (d+1)w^{\frac{d+1}{2}} \frac{\Gamma\left(\frac{k^2}{4w}+\frac{1}{2}+\frac{1}{2}\sqrt{(\frac{d+1}{2})^2+M^2L^2}\right)\Gamma\left(-\sqrt{(\frac{d+1}{2})^2+M^2L^2}\right)}{\Gamma\left(\frac{k^2}{4w}+\frac{1}{2}-\frac{1}{2}\sqrt{(\frac{d+1}{2})^2+M^2L^2}\right)\Gamma\left(\sqrt{(\frac{d+1}{2})^2+M^2L^2}\right)}$$

The brane problem at large w

- Let us consider the case of lagre w and small k.
- The field configuration is found by matching the semiclassical solution with the solution in pure Lifshitz.
  - At $z = 2$ we know the explicit solution in the Lifshitz area.
  - At different z we consider perturbations over k.

$$\Phi_0 = u^{\frac{d+z-1}{2}} J_\nu \left(\frac{\tilde{w} u^z}{z} \right), \nu = \sqrt{\frac{M^2 L^2}{z^2} + \left(\frac{d+z-1}{2z}\right)^2}$$

- The dispersion relation is

$$w = w_0 + \frac{k^2}{2w_0} + O\left(\frac{1}{u_0}\right)$$

- This formula is Lorentz-invariant at $k << w, w_0$.

Generalization for a vector field

- Let us consider a brane problem for the probe vector field at $w, k << 1/u_0$.
- We can treat it in the same way as the scalar.
- In the case of Neumann boundary conditions at both branes the dispersion relation is

$$w^2 = k^2 \frac{\int_{u_1}^{u_0} du \frac{fg}{u^{d-3}}}{\int_{u_1}^{u_0} du \frac{g}{fu^{d-3}}} \tag{1}$$

- Lorentz-invariance emerges at $d \leq 4$.

Conclusions

- We studied the solutions for gravitational and field equations describing space interpolating between *AdS* and Lifshitz at general d, z, found the conditions of stability of the solutions;
- explored correlator of scalar field in this space:
  - At small w, k the correlator is Lorentz-invariant,
  - at large w, k the correlator is the same as in pure Lifshitz,
- If the space is bounded by the brane on the *AdS* side, the excitations are Lorentz-invariant at
  - small w, k;
  - large w and small k.
- future directions:
  - generalization for the case of gauge field,
  - Search for massless Lorentz-invariant mode.