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HINTS ON INTEGRABILITY IN THE WILSONIAN/HOLOGRAPHIC RG

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I. Correspondence between D–dimensional gauge and $(D+1)$–dimensional GR theories:

$$\lim_{N\to\infty,\,\lambda\to\infty}\left\langle e^{i\,\sum_n \int d^Dx\, J_n(x)\, trO_n}\right\rangle \approx e^{i\, S^{D+1}_{min}[J_n(x)]}$$

Single trace operators $\to$ single particle states in $(D+1)$–dimensions.

Extra dimension $\to$ energy scale.

How general is this correspondence?

II. Simple example — Hermitian matrix scalar field theory:

$$\mathcal{S}[\phi] = -\frac{N}{2}\int Tr\left[\phi(p)\,(p^2+m^2)\;\; K_\Lambda^{-1}(p^2)\,\phi(-p)\right]\, d^Dp +$$
$$+N\sum_{l=0}^{\infty}\int d^Dk_1\ldots d^Dk_l\,Tr\left[\phi(k_1)\;\;\ldots\;\phi(k_l)\right]\; J_l(k_1,\ldots,k_l).$$

Cut–off — $K_\Lambda(p^2)\sim 1$ **as** $p^2 \ll \Lambda^2$**,**

$K_\Lambda(p^2)\to 0$ **as** $p^2 \gg \Lambda^2$

$J_l(k_1,\ldots,k_l) = 0$ **for all** l **and for all** $|k_l| > \lambda$**,**

λ — **low energy scale**

III. Polchinski equation:

RG invariance — $\Lambda \frac{d}{d\Lambda} e^{-W(J)} = 0$

W — **effective action**

The equation is:

$$\Lambda \frac{d\mathcal{S}_I[\phi]}{d\Lambda} = -\frac{1}{2} \int \frac{d^D p}{p^2 + m^2} \Lambda \frac{dK_\Lambda(p^2)}{d\Lambda} \times$$
$$\times \left[N^{-1} \frac{\delta^2 \mathcal{S}_I[\phi]}{\delta\phi^{ij}(-p)\delta\phi^{ji}(p)} + \frac{\delta \mathcal{S}_I[\phi]}{\delta\phi^{ij}(p)} \frac{\delta \mathcal{S}_I[\phi]}{\delta\phi^{ji}(-p)} \right],$$

where

$$\mathcal{S}_I = \sum_{l=0}^{\infty} \int d^D k_1 \ldots d^D k_l \, Tr \left[\phi(k_1) \;\; \ldots \; \phi(k_l) \right] \, J_l(k_1, \ldots, k_l),$$

and

$J_l\left(k_1, k_2, \ldots, k_l\right) = J_l\left(k_l, k_1, \ldots, k_{l-1}\right) = \ldots$ — **cyclic symmetry**

Explicitly:

$$Tr\left[\frac{\delta^2\mathcal{S}_I}{\delta\phi(p)\delta\phi(-p)}\right] = \sum_{l=1}^{\infty}\sum_{a,b=1}^{l}\int dk_1\dots d\hat{k}_a\dots d\hat{k}_b\dots dk_l \times$$
$$\times Tr\left[\phi(k_{a+1})\ \ \dots\ \phi(k_{b-1})\right]\ Tr\left[\phi(k_{b+1})\ \ \dots\ \phi(k_{a-1})\right]\times$$
$$\times J_l\left(p, k_{a+1},\dots,k_{b-1},-p,k_{b+1},\dots,k_{a-1}\right),$$

$$Tr\left[\frac{\delta\mathcal{S}_I}{\delta\phi(p)}\frac{\delta\mathcal{S}_I}{\delta\phi(-p)}\right] =$$
$$= \sum_{l,j=1}^{\infty}\sum_{a=1}^{l}\sum_{b=1}^{j}\int dq_1\dots d\hat{q}_a\dots dq_l dk_1\dots d\hat{k}_b\dots dk_j \times$$
$$\times Tr\left[\phi(q_1)\dots\phi(\hat{q}_a)\dots\phi(q_l)\ \ \phi(k_1)\dots\phi(\hat{k}_b)\dots\phi(k_j)\right]\times$$
$$\times J_l\left(p, q_{a+1},\dots,q_{a-1}\right)\ J_j\left(-p,k_{b+1},\dots,k_{b-1}\right).$$

Notations:

$$\int dk_1\dots d\hat{k}_a\dots d\hat{k}_b\dots dk_l := \int dk_{1-a-b-l},$$
$$\int dq_1\dots d\hat{q}_a\dots dq_l dk_1\dots d\hat{k}_b\dots dk_j := \int dq_{1-a-l}\int dk_{1-b-j},$$
$$J_l\left(p, k_{a+1},\dots,k_{b-1},-p,k_{b+1},\dots,k_{a-1}\right) := J_l\left(p,k_{a-b},-p,k_{b-a}\right),$$
$$J_l\left(p,q_{a+1},\dots,q_{a-1}\right) := J_l\left(p,q_{a-a}\right).$$

Higher trace operators are generated.

How to close the system of equations?

Standard Wilsonian RG — use full algebra of operators and OPE:

$O_n \, O_m = \sum_l C^l_{nm} \, O_l$ **— includes higher–trace operators.**

The result:

$\dot{J}_n = \beta_n \left(\{J\}\right)$, **dot** $= d/d\Lambda$.

However AdS/CFT–correspondence tells us that we have to consider only single trace operators!

At large N **— factorization:**

$\langle \prod_n O_n \rangle = \prod_n \langle O_n \rangle + \mathcal{O}\left(1/N^2\right)$

At large N **single trace operators form a basis through which any operator can be expressed algebraically.**

We take the quantum average of the Polchinski equation.

$\phi = \phi_0 + \varphi$, ϕ_0 **— low–energy modes ($p < \lambda$) — solution of the equations fo motion;** φ **— high–energy ($p > \lambda$).**

$\langle \dots \rangle = \int D\varphi ...$

$$\sum_{l=1}^{\infty} \int dk_{1-l}\, T_l(k_{1-l}) \dot{J}_l(k_{1-l}) = -\frac{1}{2} \int \frac{dp\, \dot{K}_\Lambda(p^2)}{p^2 + m^2} \times$$
$$\times \left[N^{-1} \sum_{l=1}^{\infty} \sum_{a,b=1}^{l} \int dk_{1-a-b-l} T_{|b-a|}(k_{a-b})\, T_{l-|a-b|}(k_{b-a}) \times \right.$$
$$\times J_l(p, k_{a-b}, -p, k_{b-a}) + \sum_{l,j=1}^{\infty} \sum_{a=1}^{l} \sum_{b=1}^{j} \int dq_{1-a-l} \int dk_{1-b-j} \times$$
$$\left. \times\, T_{l+j-2}(k_{1-a-l}, q_{1-b-j})\, J_l\left(p, q_{a-a}\right)\, J_j\left(-p, k_{b-b}\right) \right],$$

Here

$T_l(k_{1-l}) := \langle Tr[\phi(k_1) \dots \phi(k_l)] \rangle$

The equation contains only J's and T's. How to close the system of equations?

We have to add RG equations for T's:

$T_l = \frac{\delta W(J)}{\delta J_l}$ — **momentum conjugate to J_l.**

Demand RG invariance of $I(T) = \left[\int T\, J - W(J)\right]\Big|_{T=\delta W/\delta J}$.

Then we obtain Hamiltonian equations for J's and T's:

$\dot{J}_l(k_{1-l}) = \delta H(J,T)/\delta T_l(k_{1-l})$

$\dot{T}_l(k_{1-l}) = -\delta H(J,T)/\delta J_l(k_{1-l})$

Here

$$H = -\frac{1}{2}\int \frac{dp\,\dot{K}_\Lambda(p^2)}{p^2+m^2}\,\Bigg[N^{-1}\sum_{l=1}^{\infty}\sum_{a,b=1}^{l}\int dk_{1-a-b-l}\times$$

$$\times T_{|b-a|}(k_{a-b})\,T_{l-|a-b|}(k_{b-a})\,J_l(p,k_{a-b},-p,k_{b-a})+$$

$$+\sum_{l,j=1}^{\infty}\sum_{a=1}^{l}\sum_{b=1}^{j}\int dq_{1-a-l}\int dk_{1-b-j}\times$$

$$\times\, T_{l+j-2}(k_{1-a-l},q_{1-b-j})\,J_l\left(p,q_{a-a}\right)\;J_j\left(-p,k_{b-b}\right)\Bigg].$$

At low energies $\sim J^2$ terms in the Hamiltonian are suppressed wrt $\sim J$.

As well derivatives are suppressed.

Survives only:

$$\int d^D x J_l(x)\, Tr\phi^l(x) \rightarrow$$

$$\rightarrow \int d^D k_1 \dots d^D k_l\, Tr\left[\phi(k_1) \;\; \dots \, \phi(k_l)\right] \, J_l(-k_1 \dots - k_l)$$

Then, if we define:

$\Pi_n(p) = \frac{1}{N} \int_{p_1 \dots p_n} \delta(p - p_1 - \cdots - p_n) \langle Tr[\phi(p_1) \dots \phi(p_n)] \rangle$

$T = \int \frac{dp\, K_\Lambda(p^2)}{p^2+m^2}$

$J(T, \sigma, x) = \sum_k \sigma^k J_k(T, x)$

$\Pi(T, \sigma, x) = \sum_k \sigma^{-(k+1)} \Pi_k(T, x)$

The Hamiltonian reduces to:

$$H = \int_{-\pi}^{\pi} d\sigma \int d^D x \, \Pi^2 \, J'.$$

The equations of motion for $J(t, \sigma)$:

$$-\partial_t \left(\frac{\dot{J}}{J'} \right) + \frac{1}{2} \partial_\sigma \left(\frac{\dot{J}}{J'} \right)^2 = 0.$$

Inviscid Burger's or Hopf equation and is know to be integrable.